Coherent States, Kähler Manifolds and Quantum Dynamics

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Abstract. Applications of Kahler manifolds to the temporal dynamics of multilevel quantum systems in external fields are considered. It is shown by using the representation of coherent states (CS), the temporal evolution of the state vector is reduced to the “classical” dynamics of the complex parameters of the CS taking values in the coset space of the dynamical group of the Hamiltonian.

In 1974, the outstanding Soviet mathematician F.A. Berezin proposed a method for quantizing dynamical systems whose phase spaces are Kähler manifolds with a non-Euclidean metric [1]. Berezin's general approach is based on the use of complex analysis and the formalism of functional integration for the evolution operator of a quantum system, which Berezin successfully applied in the development of the quantization formalism for bosons and fermions. Complex Kähler manifolds are currently used in many areas of modern theoretical and mathematical physics.

In this talk, we will consider a compact complex Kähler manifolds associated with the dynamical groups $G$, which are interesting for some problems of quantum optics and quantum informatics.

The group-theoretical coherent state $|CS\rangle$ can be defined by the formula [2]:

$$|CS\rangle = |Z\rangle = \hat{T}(g_z)|\Psi_0\rangle,$$

where $g_z$ is an element of the Lie group $G$, corresponding to the point $g_zG_0$ for the homogeneous space $G/G_0$. The subgroup $G_0 \subset G$ conserve the initial vector $|\Psi_0\rangle$ up to a phase factor.

The evolution of the CS parameters leads to the classical dynamics for the classical analog of the quantum problem. In this case, the analog of the classical phase space is the quotient space $G/G_0$, on which the structure of a complex Kähler manifold is naturally realized [3].

If the Hamiltonian of a quantum system is linear in the generators of dynamical group $G$, then the temporal evolution of the quantum problem will be purely classical - the CS is a non-spreading wave packet moving along a classical trajectory in the corresponding generalized phase space [3]. The exact solution of the time-dependent Schrödinger equation $|\Psi(t)\rangle$ looks like:

$$|\Psi(t)\rangle = e^{-iHt/\hbar} |Z(t)\rangle,$$

where $Z(t) = (z^1(t), \ldots, z^n(t))$ is the CS trajectory in space $G/G_0$, and $\chi(t) = \int_0^t \langle Z | H' (t') | Z \rangle dt'/\hbar$.

In the more general case, it is necessary to take into account a quantum corrections, looking for solution of the Schrödinger equation in the form of superposition of the CS:
\[ |\Psi(t)\rangle = \int_{G/G_0} F(Z, Z_0 | t)|Z\rangle d\mu(Z, \bar{Z}). \]  

Here \( \lim_{t \to 0} F(Z, Z_0 | t) = \delta(Z - Z_0), \) \( |\Psi(0)\rangle = |Z_0\rangle, \) and \( \delta(Z - Z_0) \) is the \( \delta \)-function (the kernel of the unit operator in the Hilbert space of functions in the space \( G/G_0 \)) [3].

Currently, in quantum optics and quantum informatics, new unique measuring instruments have been developed that allow one to operate with one or more atoms and photons. Along with the study of quantum operations with a system of qubits, for which the group of dynamical symmetry is the group \( SU(2) \), generalizations of quantum computing and quantum communication schemes known for qubits are being actively developed, in which the elementary quantum cells will be multilevel generalizations of qubits. Therefore, the problem of studying the temporal level quantum systems interacting with each other and with external structured electromagnetic fields continues to be relevant.

In the present talk, it will be shown that using the embedding of the direct product of the dynamical group \( SU(2) \times SU(2) \) of two qubits, taking into account their dipole-dipole interaction, to the group \( SU(4) \), and makes it possible to reduce the problem for a quadratic Hamiltonian in the generators of the group \( SU(2) \times SU(2) \) to a linear realization of the Hamiltonian over the generators of the group \( SU(4) \). As a result, we have constructed a "classical" analogue of this system using the Kähler manifold \( SU(4)/U(3) \).

References

