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Riemann Hypothesis from the Physicist's Point of View

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ABSTRACT

This article presents an attempt to comprehend the evolution of the ideas underlying the physical approach to the proof of one of the problems of the century - the Riemann hypothesis regarding the location of non-trivial zeros of the Riemann zeta function. Various formulations of this hypothesis are presented, which make it possible to clarify its connection with the distribution of primes in the set of natural numbers. A brief overview of the main directions of this approach is given. The probable cause of their failures is indicated - the solution of the problem within the framework of the classical Turing paradigm. A successful proof of the Riemann hypothesis based on the use of a relativistic computation model that allows one to overcome the Turing barrier is presented. This model has been previously applied to solve another problem not computable on the classical Turing machine - the calculation of the sums of divergent series for the Riemann zeta function of the real argument. The possibility of using relativistic computing for the development of artificial intelligence systems is noted.

Keywords: Riemann zeta-function, Riemann hypothesis, non-trivial zeroes, Turing barrier, relativistic computations, horizon

1 Introduction

B. Riemann formulated his famous hypothesis in 1859. So far, it had neither a recognized proof nor a refutation. D. Gilbert in 1900 included it in the number of cardinal problems of mathematics under number 8. At present, the Clay Mathematic's Institute has included it among the seven problems of the millennium, for the solution of which a prize of 1 million US dollars will be paid.

In short, we are talking about nontrivial zeros of the Riemann zeta function [1],

$$\zeta(w) = \sum_{n=1}^{\infty} n^{-w}, w = u + iv \tag{1}$$

u, v – real numbers, more precisely about their location. According to the Riemann hypothesis (RH), they are located in the plane of complex *w* on the so-called critical line $u = ½$ [2][1].

[1] In contrast to the trivial zeros located on the real axis *w* at the points $w_n = -2n$, $n > 0$ – is an integer

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A popular historical review of the problem and directions of its investigation were given in the book [3]. Most researchers of the problem believe that the Riemann hypothesis is correct, but opposite statements are also found [4]. The papers devoted to the proof of the Riemann hypothesis can be divided into works constructing proofs using purely mathematical methods [5 - 9] and works in which its relation to various physical problems is noted [10, 11] and accordingly using physical methods of proof.

Special mention should be made of the papers of A. Turing devoted to a numerical study of the zeros of ζ-function on a classical computer. Note that Turing himself considered the Riemann hypothesis to be erroneous and his investigations were aimed at finding nontrivial zeros of the ζ -function outside the critical line. A review of the papers of A. Turing is made in the report [12].

The present work is devoted exclusively to the physical approach in search of evidence of RH. There are several reasons for this. First, the physical evidence of RH is presented below [13]. Secondly, the more than a century-long history of unsuccessful searches for the mathematical proof of RH makes us raise the question of its principal existence. For the first time, negative considerations on this subject were expressed in the report [14]. In this paper, they are further developed. And finally, the works of B. Riemann on the zeta function have direct physical consequences, which previously did not pay due attention.

2 Various formulations of RH

Before we give various formulations of RH, which make it possible to more clearly represent its contents and evaluate its significance for both mathematics and physics, we will give a quote attributed to P. Sarnak [15]

"The Riemann hypothesis is the central problem and it implies many, many things. One thing that makes it rather unusual in mathematics today is that there must be over five hundred papers - somebody should go and count - which start `Assume the Riemann hypothesis ' and the conclusion is fantastic. And those [conclusions] would then become theorems ... With this one solution, you would have proven five hundred theorems or more at once."

From this, it becomes clear that RH (even unproven) is the basis for the huge structure of mathematical theorems, erected on it as a foundation. Thus its importance for mathematics is difficult to overestimate.

Usually, in literature, the importance of RH is emphasized by the fact that it is the key to finding the distribution of primes, which is described by the Gauss function $\pi(X)$, which gives the number of primes not exceeding a real positive number X. To better understand the content of RH, we present its various formulations, in addition to the one used above [16].

i. For any real number X, the number of prime numbers less than X is approximately $Li(X)$ and this approximation is essentially square root accurate[2]
   It is stated here that for sufficiently large X >>1 the error $|\pi(X) - Li(X)| < (X)^{0.5+\alpha}$, where a number $\alpha > 0$ is arbitrarily small.

ii. All the nontrivial zeroes of ζ(s) lie on the vertical line in the complex plane consisting of the

[2] $Li(x) = \int_2^x \frac{dt}{\ln t}$ - integral logarithm

complex numbers with real part equal to ½. These zeroes are none other than ½ ± iϑ_1, ½ ± iϑ_2, ½ ± iϑ_3, … where ϑ_1, ϑ_2, ϑ_3, … comprise the spectrum of primes.

Some other formulations of RH are given in [16]. Let us dwell in detail on the second formulation, which belongs to B. Riemann. It allows us to state that using the Fourier transform, knowing the non-trivial zeros of the zeta function, we can construct a regular function that describes the distribution of primes arbitrarily accurately. This is the subject of the pioneering work of B. Riemann [2]. A description of this procedure is given in [16].

Given the role that prime numbers play in mathematics, the value of this result can hardly be overestimated. Interestingly, mathematicians evaluate this result in the following words [16]:

> ”This infinite collection of complex numbers, i.e., the nontrivial zeroes of the Riemann zeta function, plays a role with respect to $\pi(X)$ rather like the role the spectrum of the Hydrogen atom plays in Fourier's theory. Are the primes themselves no more than an epiphenomenon, behind which there lies, still veiled from us, a yet-to-be-discovered, yet-to-be-hypothesized, profound conceptual key to their perplexing orneriness? Are the many innocently posed, yet unanswered, phenomenological questions about numbers such as in the ones listed earlier waiting for our discovery of this deeper level of arithmetic? Or for layers deeper still? Are we, in fact, just at the beginning?” [3]

From the point of view of physics, we note that Riemann, long before H. Poincare, who discovered the complex behavior of simple dynamical systems, showed the possibility of describing a random function (π (X)) using iterations in which only regular deterministic functions participate.

3 Attempts to prove RH by physical methods

Since this issue is sufficiently fully described in the literature (see, for example, [17], we restrict ourselves to a brief statement of the main points, following [18][4]:

> “Remarkably, it turns out that striking similarities exist between the Riemann zeros and the quantum energy levels of classically chaotic systems Michael Berry pointed out, in the mid-1980s, that Riemann’s formula relating the zeros to the primes is closely analogous to one discovered by Martin Gutzwiller in 1971 that relates quantum energy levels to unstable classical periodic orbits in the semi-classical limit[19 – 21]. In 1973, following a comment by Freeman Dyson, Hugh Montgomery conjectured that correlations between the heights of the zeros on the scale of their mean spacing are statistically the same as those of the eigenvalues of large random Hermitian matrices. And quantum spectral statistics in classically chaotic systems are also believed to match the predictions of random-matrix theory. These observations hint at the possibility that the heights of the Riemann zeros might be the energy levels of some quantum chaotic system. Interestingly, if this were true, it would prove the Riemann hypothesis! The connections between the Riemann hypothesis and quantum mechanics have generated considerable excitement. But

[3] Perhaps the authors wanted to say that the role of the Riemann hypothesis and, especially, its proof will play in mathematics the same role that the explanation of the spectrum of the hydrogen atom played in physics, laying the foundation for the successful development of quantum mechanics.

[4] A popular summary of these issues is given in [3]

although such connections have inspired several new lines of attack, a proof still continues to be elusive. Nevertheless, these connections have been used to suggest answers to some other long-standing and important problems relating to the zeta function."

A more detailed collection of links can be found in the work [17]. It should be recognized that despite the efforts expended, physical analogies did not allow obtaining final evidence of RH.

Concluding this section, we also mention the report of M. Atiyah at the Heidelberg Laureate Forum, 23 – 28 Sept 2018, in which he presented his evidence for RH. This work is not directly related to physics. The author builds the proof based on the properties of the Todd function $T(s)$ introduced by him, one of which is that on the critical line it behaves as follows

$$\lim_{y\to\infty} T(\frac{1}{2}+iy)=\frac{1}{\alpha},\alpha=\frac{e^2}{\hbar c}\approx\frac{1}{137} \quad (2)$$

where α – is a fine structure constant, $e, \hbar, c$ – an electron charge, Plank constant and the light speed [22]. After numerous discussions widely reflected on the Internet, the scientific community has not recognized the evidence as true[5].

4 The proof of the Riemann hypothesis

The calculation of the zeta function of the complex argument and the proof the RH [13] are based on the study of the behavior of the *m*-th partial sums of the series (1)

$$\zeta_m(w)=\sum_{n=1}^{m} n^{-w}, w=u+iv \quad (3)$$

It is known that the values of successive partial sums of series (1) describe asymptotically the vortex trajectory in the complex plane [23] –see Fig. 1

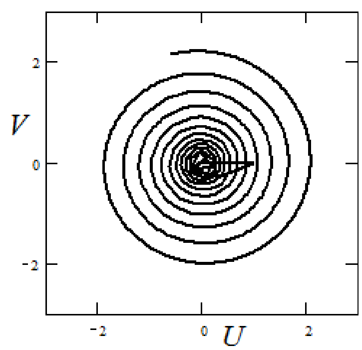

Figure 1. Calculation of $\zeta_m(s_1)$; s_1=0.5-14.134725i - first non-trivial zero of zeta function; U = Re [$\zeta_m(s_1)$], V = Im [$\zeta_m(s_1)$]; 1<m< 1001

[5] An analysis of Atiyah's arguments can be found at https://rjlipton.wordpress.com/2018/09/26/reading-into-atiyahs-proof/

Let us treat the relativistic dynamics of a material point on a plane whose trajectory is depicted in Fig. 1. Consider a flat metric in the resting coordinate system r', φ', z', t' (*x, y* - Cartesian coordinates)

$$ds'^2 = c^2 dt'^2 - dr'^2 - r'^2 d\varphi'^2 - dz'^2$$
$$r' = \sqrt{x^2 + y^2}, \varphi' = arctg\frac{y}{x} \tag{4}$$

Perform a transition to a coordinate system in which the vortex rests

$$dr' = \alpha(r,t)\left(dr + \frac{\delta}{r}dt\right),$$
$$d\varphi' = d\varphi + \frac{\omega}{r^2}dt, \tag{5}$$
$$t' = t, z' = z$$

where *δ* and *ω* are constants dependent on the parameters of a vortex, and *α(r,t)* is chosen from the condition that the first expression (5) represents a complete differential. If we choose *α(r) = cr/ω* and perform the conversion

$$d\varphi dt = \frac{1}{r'_\varphi} dr dt, r'_\varphi = \frac{dr}{d\varphi}$$
$$dr^2 + r^2 d\varphi^2 = dl^2 \tag{6}$$

where *dl* – is length element along the line representing the projection of the three-dimensional vortex trajectory on the plane *z = const* and by a suitable coordinate transformation eliminate the terms ~ *drdt*, we get the final expression for the metric in own system of a vortex

$$ds^2 = A(r)(cdt)^2 - dl^2 + B(r)dr^2,$$
$$A(r) = b^2 - \left(\frac{\tilde{r}}{r}\right)^2, b^2 = 1 - \left(\frac{\delta}{\omega}\right)^2, \tilde{r} = \frac{\omega}{c}, \tag{7}$$
$$B(r) = 1 - \left(\frac{r}{\tilde{r}}\right)^2 - \left(\frac{\delta}{\omega}\frac{r}{\tilde{r}} + \frac{\tilde{r}}{r'_\phi}\right)^2 A^{-1}$$

Further, relativistic equations of motion are derived from (7) (Γ_{kl}^i – Christoffel symbols [24], *i,k,l* = 0,1,2)

$$\frac{d^2x^i}{ds^2} + \Gamma_{kl}^i \frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$
$$x^0 = ct, x^1 = r, x^2 = l \tag{8}$$

Metric (7) has a singularity that determines the so-called event horizon that is reached in infinite time according to the watch of the rest observer. To describe the behavior of the solution under the horizon, one usually performs the transformation to the Finkelstein coordinates [24] (the ± signs correspond to the movement to the center *r* = 0 and from it)

$$dr = \frac{f \cdot (d\rho \pm cd\tau)}{\sqrt{-A(r)B(r)}}, cdt = \frac{f^2 d\rho \pm cd\tau}{1 - f^2} \tag{9}$$

where the function *f* is chosen from the condition for eliminating the fictitious singularity of the metric (7) on the horizon. This is achieved by

$$f(r) = [1 - A(r)]^{1/2} = \left[\left(\frac{\delta}{\omega}\right)^2 + \left(\frac{\tilde{r}}{r}\right)^2\right]^{1/2} \tag{10}$$

Factually, this way does not in agreement with the instability of the electromagnetic vacuum under the event horizon [25], which makes it impossible to propagate signals. Sending for details to [13], we note that to take advantage of the developed formalism using the Finkelstein coordinates, it suffices to set dτ = 0 in formulas (9).

Solving the equations of motion (8) below and above the horizon and stitching them, we can conclude that on the critical line *u* = ½ in the complex plane there are two types of solutions - I and II, which correspond to the zeros of the zeta function (I) and its nonzero values (II) – see Fig. 2.

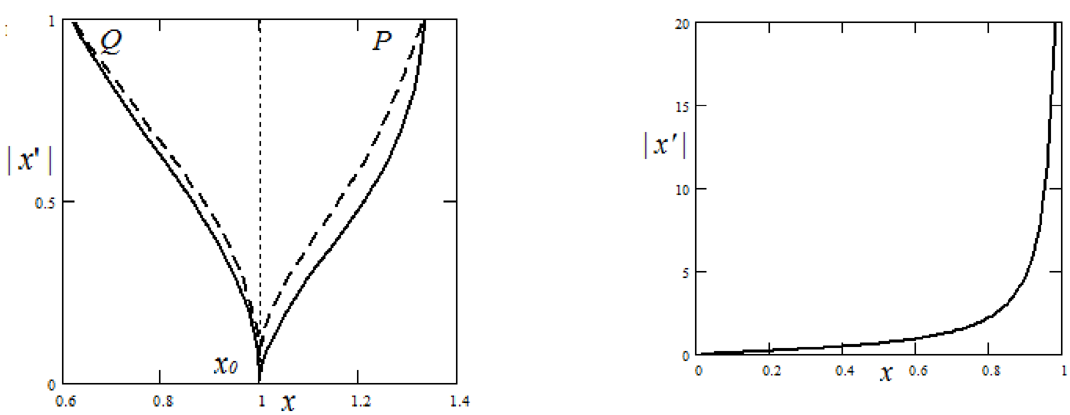

Figure. 2. The phase portraits of the solutions of equations of motion (8): right –type *I*, left – type *II*. $x = r/\tilde{r}, x' = dx/ds$ **. The solid and the dashed lines show the two branches of the solution, dotted line – the horizon x_0; *P, Q* -are the branch points, where $r_s' = \pm 1$.**

At the same time, outside the critical line, only type II solutions exist, which indicates the absence of zeros of the zeta function. This proves the Riemann hypothesis [13].

5 Discussion

The question of the validity of the proof of any hypothesis, and even more so the one such as RH, always causes numerous discussions. Moreover, the present time is rich in publications devoted to this issue. To verify this, one can visit the website of the Cornell University electronic library arxiv.org. However, there are no discussions on this topic. The only exception is the report of M. Atiyah, which is mentioned above.[6] John Derbyshire tried to highlight the reasons for this decline in interest in the subject under discussion in his book [3]. He's writing:

> "There are breakthroughs followed by outbursts of enthusiasm, and there are stalemates followed by spells of apathy. My impression of the current (mid-2002) state of affairs— though, to be sure, it is only the impression of a noncombatant—is that researchers are stalemated. We are in a lull. The great burst of interest generated by Deligne's proof of the Weil Conjectures in 1973 and by the Montgomery-Odlyzko developments of 1972−1987 seems to me to have spent itself…. Sir Michael Berry, who has a way with words, has coined the concept of the "clariton," which he defines to be "the elementary particle of sudden understanding." In the realm of the RH, claritons are currently in short supply."

This was written in the year the book was published [3] - 2003. What happened in the following years did not at all contribute to a revival of interest in RH. Here is an excerpt from a letter from M. Berry dated September 5, 2019, to the author about the work [13]:

> "The Riemann hypothesis is mathematics, so it cannot be proved using physics. Your relativistic analysis is an unusual formalism, and people are probably unwilling to make the effort to follow your arguments. But any formalism can be transformed into any other, so it would help readers if you translate your arguments into a more conventional mathematical language".

Perhaps this is evidence only of a disappointment in the possibility of proving RH by physical methods.

From a brief review of early physical works devoted to the search for evidence of the Riemann hypothesis, it can be seen that their failures were most likely because the authors hoped that its physical analogs were easier to solve, which turned out to be far from the truth.

A reason for the successful proof of RH in [13] was its connection with overcoming the Turing barrier in calculating the roots of the Riemann zeta function due to relativistic effects. The term "relativistic computer" (RC) first appeared in the work of I. Nemeti (with co-authors) in 1987 [26]. It was supposed to use relativistic effects to overcome the limitations imposed on the calculations carried out in the framework of the model of the classical Turing machine (MT) - "Turing barrier". In particular, it was suggested that the RC can be used to solve the so-called non-computable tasks that cannot be solved on classical MT (requires infinite time). For this, it was proposed to use the achievements of relativistic physics - Kerr-Newman black holes, the space-time metric near which has singularities that ensure the natural adopting of infinity in the number of allowed computation time values. However, since the implementation of such projects to this day remains problematic, they have no practical consequences. In the works of the author [27, 28], the idea of the RC received a practical embodiment in the form of

[6] A little earlier, there were reports of the work of the Nigerian mathematician Opeyemi Enoch in which errors were indicated in the proof.

developing an algorithm for calculating the sums of divergent series — a task also related to the problems which are non-computable on MT.

It can be assumed that the failure of previous attempts to prove the Riemann hypothesis, both purely mathematical and involving physical methods, the equivalence of which is indicated by Sir M. Berry is since they were carried out within the framework of the Turing paradigm.

Very interesting is the ability to use the results obtained to unravel the intellectual abilities of the brain, which are believed to be based on the use of non-computable algorithms [29]. In this regard, the use of relativistic calculations is a very promising direction, since they naturally include phenomena associated with the curvature of space-time i.e. gravity [30]. Of particular interest are these results for the development of artificial intelligence systems.

6 Conclusion

This article presents an attempt to comprehend the evolution of the ideas underlying the physical approach to the proof of one of the problems of the century - the Riemann hypothesis regarding the location of non-trivial zeros of the Riemann zeta function. Various formulations of this hypothesis are presented, which make it possible to clarify its connection with the distribution of primes in the set of natural numbers. A brief overview of the main directions of this approach is given. The probable cause of their failures is indicated - the solution of the problem within the framework of the classical Turing paradigm. A successful proof of the Riemann hypothesis based on the use of a relativistic computation model that allows one to overcome the Turing barrier is presented. The possibility of using relativistic computing for the development of artificial intelligence systems is noted.

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