Geometrization of Maxwell's Equations in Extended Riemannian Geometry

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Constitutive tensor

$$F = \frac{1}{2} F_{\underline{\alpha\beta}} \, \mathrm{d}x^{\underline{\alpha}} \wedge \mathrm{d}x^{\underline{\beta}}, \quad F \in \Lambda^2,$$
$$G = \frac{1}{2} G^{\underline{\alpha\beta}} \partial_{\underline{\alpha}} \wedge \partial_{\underline{\beta}}, \quad G \in \Lambda_2,$$
$$j = j^{\underline{\alpha}} \partial_{\underline{\alpha}}, \quad j \in \Lambda_1.$$
$$G^{\alpha\beta} = \lambda(F_{\gamma\delta}).$$

Maxwell equations (exterior calculus):

$$\mathrm{d}F = 0, \qquad \delta G = \frac{4\pi}{c} j, \tag{1}$$

Divergence δ through the Hodge duality operator:

$$* : \Lambda^k \to \Lambda^{n-k},$$
$$\delta = (-1)^k *^{-1} \mathrm{d} *,$$

Divergence through the Poincaré duality operator:

$$\sharp : \Lambda^k \to \Lambda_{n-k},$$
$$\delta = (-1)^k \sharp^{-1} \mathrm{d} \sharp.$$

Let's write the constitutive equations as follow:

$$G = \lambda(F).$$

Then the equation (1) takes the form:

$$\mathrm{d} \sharp \lambda(F) = \frac{4\pi}{c} \sharp j. \tag{2}$$

In addition, let us obtain the Hodge duality operator without an explicit metric specification. For this we define the isomorphism:

$$* : \Lambda^2 \to \Lambda^2, \qquad * : F \mapsto \sharp \lambda(F). \tag{3}$$

Then the equation (2) takes the form:

$$\mathrm{d}^* F = \frac{4\pi}{c} \sharp j,$$

and the operator (3) is the Hodge duality operator, defined not via the Riemannian metric, but through the functional λ.

Constitutive tensor in linear media

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}.$$

$\lambda^{\alpha\beta\gamma\delta}$ has the following symmetry:

$$\lambda^{\alpha\beta\gamma\delta} = \lambda^{[\alpha\beta][\gamma\delta]}.$$

The tensor $\lambda^{\alpha\beta\gamma\delta}$ can be represented in the following form:

$$\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{\alpha\beta\gamma\delta} + {}^{(2)}\lambda^{\alpha\beta\gamma\delta} + {}^{(3)}\lambda^{\alpha\beta\gamma\delta},$$
$${}^{(1)}\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{([\alpha\beta][\gamma\delta])},$$
$${}^{(2)}\lambda^{\alpha\beta\gamma\delta} = {}^{(2)}\lambda^{[[\alpha\beta][\gamma\delta]]},$$
$${}^{(3)}\lambda^{\alpha\beta\gamma\delta} = {}^{(3)}\lambda^{[\alpha\beta\gamma\delta]}.$$

Let's write out the number of independent components:

- $\lambda^{\alpha\beta\gamma\delta}$ has 36 independent components,
- ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ has 20 independent components,
- ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ has 15 independent components,
- ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ has 1 independent component.

Usually only part ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ is considered, since ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ and ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ make it impossible to record the electromagnetic field Lagrangian:

$$L = -\frac{1}{16\pi c} F_{\alpha\beta} G^{\alpha\beta} \sqrt{-g} - \frac{1}{c^2} A_\alpha j^\alpha \sqrt{-g}. \tag{4}$$

That is, when we use parts ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ and ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$, the tensor $F_{\alpha\beta}$ must be self-anticommute.

Riemannian geometrization of Maxwell equations

The Lagrangian of the electromagnetic field (4) we will write in the form of a Lagrangian of Yang-Mills:

$$L = -\frac{1}{16\pi c} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \sqrt{-g} - \frac{1}{c^2} A_\alpha j^\alpha \sqrt{-g}.$$

The tensor $\lambda^{\alpha\beta\gamma\delta}$ structure:

$$\lambda^{\alpha\beta\gamma\delta} = 2\sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} = \\ = \sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}\right) + \sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right).$$

Then by taking into account the symmetry of tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$:

$$G^{\alpha\beta} = \frac{1}{2}\sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right) F_{\gamma\delta}.$$

For clarity, we will write out this equation in components:

$$G^{0\underline{i}} = \sqrt{-g}\left(g^{00} g^{\underline{ij}} \smile g^{0\underline{i}} g^{0\underline{j}}\right) F_{0\underline{j}} + \sqrt{-g}\left(g^{0\underline{j}} g^{\underline{ik}} \smile g^{0\underline{k}} g^{\underline{ij}}\right) F_{\underline{jk}},$$
$$G^{\underline{ij}} = \sqrt{-g}\left(g^{\underline{i}0} g^{\underline{jk}} \smile g^{0\underline{j}} g^{\underline{ik}}\right) F_{0\underline{k}} + \sqrt{-g}\left(g^{\underline{ik}} g^{\underline{jl}} \smile g^{\underline{il}} g^{\underline{jk}}\right) F_{\underline{kl}}.$$

Let us express equations through the field vectors E_i, B^i, D^i, H_i:

$$D^{\underline{i}} = -\sqrt{-g}\left(g^{00} g^{\underline{ij}} \smile g^{0\underline{i}} g^{0\underline{j}}\right) E_{\underline{j}} + \sqrt{-g}\varepsilon_{\underline{klj}} g^{0\underline{k}} g^{\underline{il}} B^{\underline{j}},$$
$$H_{\underline{i}} = \sqrt{-g}\varepsilon_{\underline{mni}} \varepsilon_{\underline{klj}} g^{\underline{nk}} g^{\underline{ml}} B^{\underline{j}} + \sqrt{-g}\varepsilon^{\underline{klj}} g_{0\underline{k}} g_{\underline{il}} E_{\underline{j}}.$$

Permittivity ε^{ij}:

$$\varepsilon^{\underline{ij}} = -\sqrt{-g}\left(g^{00} g^{\underline{ij}} \smile g^{0\underline{i}} g^{0\underline{j}}\right).$$

Permeability μ^{ij}:

$$(\mu^{-1})_{\underline{ij}} = \sqrt{-g}\varepsilon_{\underline{mni}} \varepsilon_{\underline{klj}} g^{\underline{nk}} g^{\underline{ml}}.$$

Thus, the geometrized constitutive equations in the components have the following form:

$$D^i = \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j,$$
$$H_i = (\mu^{-1})_{ij} B^j + {}^{(2)}\gamma^j_i E_j,$$
$$\varepsilon^{\underline{ij}} = -\sqrt{-g}\left(g^{00} g^{\underline{ij}} \smile g^{0\underline{i}} g^{0\underline{j}}\right),$$
$$(\mu^{-1})_{\underline{ij}} = \sqrt{-g}\varepsilon_{\underline{mni}} \varepsilon_{\underline{klj}} g^{\underline{nk}} g^{\underline{ml}},$$
$${}^{(1)}\gamma^i_j = {}^{(2)}\gamma^i_j = \sqrt{-g}\varepsilon_{\underline{klj}} g^{0\underline{k}} g^{\underline{il}}.$$

The following equation is valid:

$$\varepsilon^{\underline{ij}} = \mu^{\underline{ij}}$$

under the condition $g^{0\underline{i}} = 0$. This means that the geometrization of Maxwell's constitutive equations on the basis of a quadratic metric imposes a restriction on the impedance:

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = 1.$$

This result is a consequence of the insufficient number of components of the Riemannian metric tensor $g_{\alpha\beta}$ (10 components), even for tensor ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ (20 components), not to mention the total tensor $\lambda^{\alpha\beta\gamma\delta}$ (36 components). Even the usage of the geometrization of Riemannian geometry with torsion and nonmetricity does not change the situation. Actually, when geometriztion is based on the Riemannian geometry, we the refractive index n_{ij}, but not the permittivity ε_{ij} and the permeability μ_{ij}.
The authors suggest that in order to solve the problem of the geometrization of the Maxwell's equations one need to rely on Finsler geometry.

$$\mathrm{d}s^4 = g_{\alpha\beta\gamma\delta} \, \mathrm{d}x^\alpha \, \mathrm{d}x^\beta \, \mathrm{d}x^\gamma \, \mathrm{d}x^\delta \, .$$

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